

Dynamics of the breakdown of granular clusters

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Recently van der Meer *et al.* studied the breakdown of a granular cluster [Phys. Rev. Lett. **88**, 174302 (2002)]. We reexamine this problem using an urn model, which takes into account fluctuations and finite-size effects. General arguments are given for the absence of a continuous transition when the number of urns (compartments) is greater than two. Monte Carlo simulations show that the lifetime of a cluster τ diverges at the limits of stability as $\tau \sim N^{1/3}$, where N is the number of balls. After the breakdown, depending on the dynamical rules of our urn model, either normal or anomalous diffusion of the cluster takes place.

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I. INTRODUCTION

Dissipation of kinetic energy during inelastic collisions in gaseous granular systems has profound consequences [1,2]. One of the most spectacular ones is the formation of spatial inhomogeneities [3], which drastically contrast with a uniform distribution of molecules or atoms whose dynamics is essentially elastic.

Some time ago Schlichting and Nordmeier presented a simple experiment that demonstrates some consequences of inelasticity of granular systems [4]. They used a container separated into two equal compartments by a wall that has a narrow horizontal slit at a certain height. The container is filled with balls (plastic or metallic) and subjected to vertical shaking. For vigorous shaking the balls distribute equally between two compartments. However, when the shaking is sufficiently mild, a nonsymmetric distribution occurs. In such a case the compartment with majority of balls, due to numerous inelastic collisions, is effectively cooler than the other one. Consequently, less balls are leaving this compartment, which stabilizes such an asymmetric distribution of balls. To explain this experiment, Eggers derived a phenomenological equation for the flux $F(n)$ of balls leaving a given compartment [5]

$$F(n) = Cn^2 \exp(-Bn^2). \quad (1)$$

In the above equation n is the concentration of balls in a given urn and B and C are constants that depend on the properties of balls, typical sizes of the system, and of parameters of shaking (the constant C may be eliminated by an appropriate redefinition of the time scale). In agreement with experiment, Eq. (1) predicts for sufficiently large B unequal distribution of balls. The above experiment was repeated in the case when the number of compartments L was greater than two by van der Weele *et al.* [6]. In such a case formation of unequal distribution of balls is accompanied by strong hysteresis that is in agreement with theoretical analysis [7]. Moreover, certain aspects of these phenomena for $L=2$ were approached using hydrodynamic equations [8].

Recently, van der Meer *et al.* examined the case of $L > 2$ further [9]. In particular, they studied dynamics of configurations (clusters) starting from all balls localized in a single compartment. Using a theoretical model based on Eq. (1), they have shown that when shaking is strong enough such a cluster breaks down and diffuses with the anomalous diffusion exponent $1/3$ [in the following we refer to this model as MWL (Meer, Weele, and Lohse)]. For less vigorous shaking, the cluster remains relatively stable and only after some time it abruptly breaks down. Some of their predictions were confirmed experimentally.

In the framework of the MWL model it is rather difficult to include the effect of fluctuations. Such fluctuations might originate due to, for example, a finite number of balls and especially close to critical points they might play an important role. In an attempt to take such effects into account a generalization of Ehrenfest's [10] urn model was recently examined in the case $L=2$ [11]. Relative simplicity of the model allows for a detailed study of its various characteristics.

The motivation of the present paper is to reexamine the breakdown of granular clusters using the urn model in the case $L > 2$. In Sec. II we define the model and present its steady-state phase diagram for $L=3$. We also argue that, in analogy to the Potts model in the mean-field limit, there are no continuous transitions for $L > 2$. In Sec. III we examine dynamics of the breakdown of clusters in a similar way as van der Meer *et al.* [9]. Although qualitatively our results are similar to theirs, in our model the diffusion of the cluster is normal with the exponent $1/2$. Moreover, we calculate the size dependence of the lifetime of a cluster τ and show that at the limits of stability it scales as $N^{1/3}$. In Sec. IV we present a modified version of the urn model that in the steady state reproduces the flux (1). The diffusion of the broken-down cluster is then shown to be anomalous with exponent $1/3$, as it was already found [9]. It was suggested that essential features of the MWL model are independent on the precise form of the flux (1), as long as it has a single hump [9]. On the contrary, our results show that at least the diffusion exponent depends on some details of the flux and not only on its qualitative shape (in our models the flux is also a single hump function). Section V contains our conclusions.

II. MODEL AND ITS STEADY-STATE PROPERTIES

Our model is a straightforward generalization of the two-urn case [11]: N particles are distributed between L urns and the number of particles in i th urn is denoted as N_i ($\sum_{i=1}^L N_i = N$). Urns are connected through slits sequentially: i th urn is connected with $(i-1)$ th and $(i+1)$ th. Moreover, periodic boundary conditions are used, i.e., first and L th urns are connected. Particles in a given urn (say i th) are subject to thermal fluctuations and the temperature T of this urn depends on the number of particles in it as

$$T(n_i) = T_0 + \Delta(1 - n_i), \quad (2)$$

where n_i is a fraction of the total number of particles in a given urn ($n_i = N_i/N$) and T_0 and Δ are positive constants. Equation (2) is the simplest function that reproduces the fact that due to inelastic collisions between particles, their effective temperature decreases as their number in a given urn increases. Next, we define the dynamics of the model as follows.

(i) One of the N particles is selected randomly. (ii) With probability $\exp[-1/T(n_i)]$ the selected particle is placed in a randomly chosen neighboring urn, where i is the urn of a selected particle.

The above rules implies that the flux of particles leaving i th urn is, up to a proportionality constant, given by

$$F(n_i) = n_i \exp\left[-\frac{1}{T(n_i)}\right], \quad (3)$$

where $T(n_i)$ is defined in Eq. (2). Let us notice that the flux (3), similarly to Eq. (1), is a single hump function. Having an expression for the flux we can write the equations of motion as

$$\frac{dn_i}{dt} = \frac{1}{2}F(n_{i-1}) + \frac{1}{2}F(n_{i+1}) - F(n_i), \quad (4)$$

where $i=1,2,\dots,L$. Steady-state properties of this model can be obtained using similar analysis as in the $L=2$ case [11] or as for $L>2$ but with fluxes given by Eq. (1) [7]. The results of this analysis in the $L=3$ case are presented in Fig. 1. In region I and II the symmetric phase ($n_1=n_2=n_3=1/3$) is stable. The continuous line in Fig. 1, which locates the limit of stability of this phase, is given by the following equation:

$$T_0 = \sqrt{\frac{\Delta}{3}} - \frac{2\Delta}{3}. \quad (5)$$

This equation has a very similar form to the corresponding equation in the $L=2$ case [11]. Asymmetric solution, where one of the urns has the majority of balls and remaining two urns have only a small equal fraction of balls ($n_1 > n_2 = n_3$), is stable in region II and III. The line separating regions I and II can be determined only numerically as a solution of a transcendental equation, similarly to the $L=2$ case [11]. There is also a third type of solution where two urns

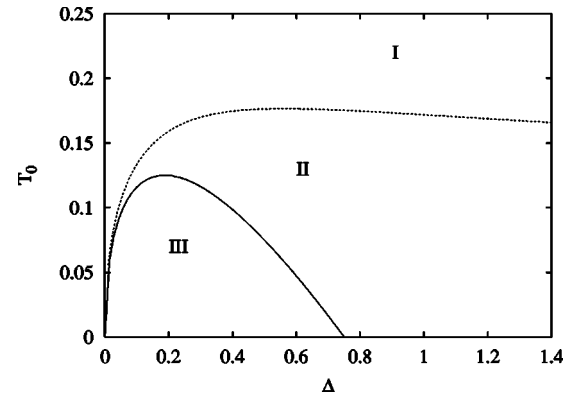


FIG. 1. The steady-state phase diagram for the three-urn model. See text for a description of phases.

contain majority of balls and the third urn has only a small fraction of them ($n_1=n_2>n_3$). Such a solution, which has saddlelike stability, exists only in region III. Similar solutions can be found for the MWL model [6,7]. An important, qualitative difference with the case $L=2$, is that regions I and III are always separated by region II where both symmetric and asymmetric solutions are stable, hence the tricritical point is located at the origin $T_0=\Delta=0$. It means that a phase transition between these two phases is always accompanied by hysteresis effects. On the other hand, in the $L=2$ case continuous transitions are possible, which are not accompanied by hysteresis [11]. Such a behavior is actually in agreement with experimental data and with MWL model [6].

Has this qualitative difference a more general explanation or is it rather a coincidental property? In our opinion, absence of continuous transitions for $L>2$ is a generic property of such systems and at least to some extent could be understood. First, let us notice that the phase transition for $L=2$ is a manifestation of the spontaneous symmetry breaking in the system: in certain regime one of the two identical urns is preferentially filled with balls. Such a situation resembles the phase transition in the $S=1/2$ Ising model, where below certain temperature the up-down symmetry is broken and the system acquires spontaneous magnetization [12]. Actually, this analogy can be confirmed more quantitatively. We have shown that for $L=2$ and at the critical point the probability distributions has the same moment ratios as in the Ising model in dimension d greater than the so-called upper critical dimension ($d>4$) [13]. Let us notice, that in our model balls are selected randomly, which means that this is essentially a mean-field model. Moreover, our model is a dynamical, spaceless model, contrary to the Ising model, which is a lattice equilibrium model. The fact that such different models have some similarities shows that as far as the critical behavior is concerned what really matters is symmetry. In both cases this is the Z_2 symmetry that is broken below the critical point.

Pushing this analogy further, we expect that for $L>2$ the phase transition in our model should be similar to the phase transition of the L -state Potts model above the critical dimension [14]. In the L -state Potts model at sufficiently low temperature one of the L symmetric ground states is preferen-

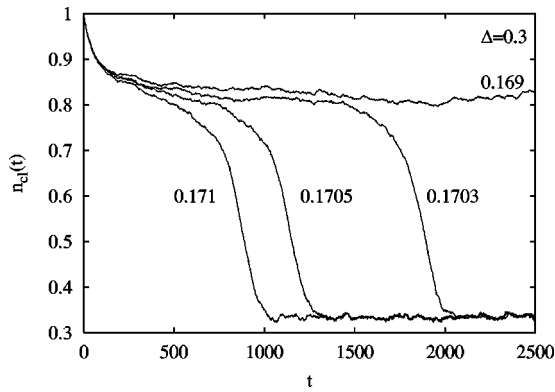


FIG. 2. The time evolution of the fraction of balls of the cluster n_{cl} close to the limits of stability of the asymmetric phase ($N=5 \times 10^4$, $L=3$). The values of T_0 are indicated. For $\Delta=0.3$ the limit of stability of the asymmetric phase is at $T_0 = 0.169829772\dots$. For a larger number of balls N , stochastic fluctuations will diminish.

tially selected. However, it is well known that above the upper critical dimension and for $L > 2$ there are only discontinuous transitions in the Potts model [14]. Consequently, the transition in the urn model, and most likely in related models, which preserves Z_L symmetry of compartments, should be discontinuous.

Let us notice that one can easily break the symmetry of the compartments, e.g., changing the boundary conditions, which in our analogy introduces some asymmetry in the Potts model. It is possible that in such a case the system effectively will become similar to the $L=2$ system and will exhibit a continuous transition. Finally, we expect that for $L > 3$ the phase diagram should be topologically similar to the one for $L=3$ shown in Fig. 1.

III. DYNAMICAL PROPERTIES OF CLUSTER CONFIGURATIONS

In the present section we study certain dynamical properties of cluster configurations. We used Monte Carlo simulation. Since it is rather straightforward, we omit a more detailed description of the numerical implementation of the dynamical rules of our model. Initially, we place all balls in one urn and examine its subsequent evolution. If the parameters T_0 and Δ are such that the system is in region I then such a cluster is unstable and after some time due to fluctuations it breaks down and balls spread throughout all urns. This is illustrated in Fig. 2 that shows the concentration of balls in the urn in which the balls were initially placed. Let us notice that (i) the breakdown is relatively abrupt and during the evolution up to the breakdown the concentration of balls only slightly decreases; (ii) upon approaching the line separating regions I and II the lifetime of the cluster τ increases. Since in region II the asymmetric state has an infinite lifetime it means that τ must diverge upon approaching this region. This behavior is seen in Fig. 3. In addition to the three-urn case we also made analogous measurements of τ for $L=5$ and 7 and the results are also shown in Fig. 3. Let

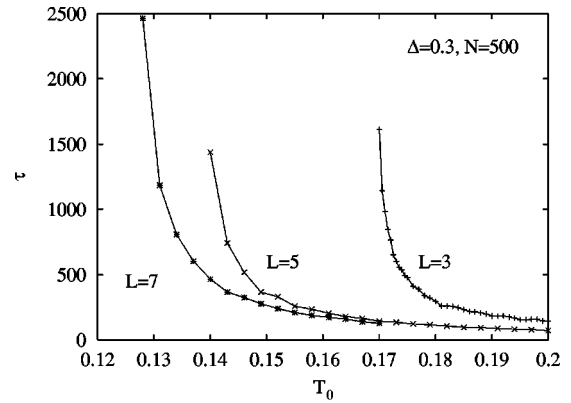


FIG. 3. The average lifetime of a cluster τ as a function of T_0 for different number of urns L . Each point is an average of at least 300 runs.

us notice that results presented in Figs. 2 and 3 are similar to those obtained by van der Meer [9], although they are parametrized by a different variable.

The limit of stability of the asymmetric phase can be regarded as a critical point. Thus, we expect that exactly at this point, e.g., the lifetime τ has a power-law divergence $\tau = N^z$, and $z > 0$. Such a behavior is shown in Fig. 4. From the slope of the straight line, which is a least-square fit to our data we estimate $z=0.32(3)$. Let us notice that in the two-urn model at the limits of stability τ exhibits a very similar divergence [11]. In the case $L=2$ more precise calculations were possible strongly suggesting that $z=1/3$ that is also consistent with the present three-urn model result. Let us emphasize that because in our model the number of balls is finite, we can study size dependent quantities as shown in Fig. 4. Such calculations would not be possible for models solely based on steady-state equations.

Finally, let us examine the breakdown of a cluster in the many-urn case $L \gg 1$. In such a case a continuous approach to the MWL model shows that after breaking down, the cluster diffuses with the anomalous exponent $1/3$ [9]. Results of our simulations are shown in Fig. 5. From these data we conclude that spreading of a cluster occurs with the ordinary

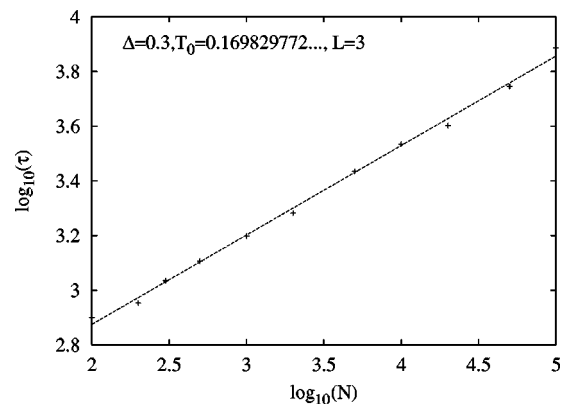


FIG. 4. The average lifetime of a cluster τ as a function of the number of balls N at the limits of stability of the asymmetric phase. Each point is an average of at least 300 runs.

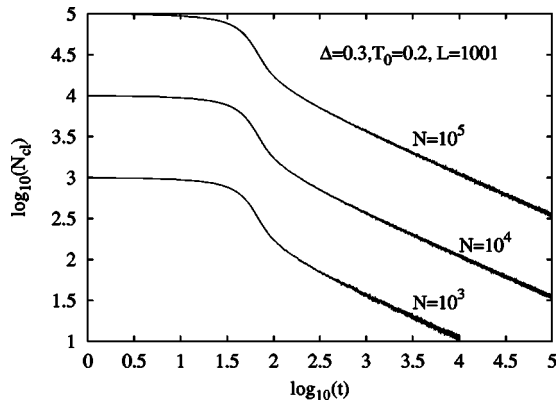


FIG. 5. The average occupancy of a central urn N_{cl} as a function of time t . The slope of decay is very close to 0.5 that confirms the diffusive nature of spreading ($N_{cl} \sim t^{-1/2}$). Each curve is obtained from averaging over 50 independent runs.

exponent 1/2 rather than anomalously. Ordinary diffusion in our model can be also easily explained analytically applying basically the same continuous approach as used in Ref. [9]. In this approach the set of equations of motion (4) is transformed into a partial differential equation. Then, one immediately realizes that the linear term in front of the exponent in Eq. (3) leads to the ordinary diffusion equation. On the other hand, the anomalous diffusion of MWL model can be traced back to the quadratic (in n) term in the flux in Eq. (1). This quadratic term is related with two-particle collisions [9].

IV. THE PAIR MODEL

One can easily construct urn models for which the expression for the flux will have a different form. In particular, redefining the effective temperature (2) and drawing each time a pair of balls we obtain an urn model with the flux of exactly the same form as Eq. (1). This dynamics takes into account some of the two particles correlations. It allows us to recover some properties of the MWL model and establish further results.

The model, which we call a pair model, is similar to the previously described one, except that its dynamics is now defined as follows. (i) Two different balls are selected randomly. (ii) If and only if the two balls are in the same urn, with probability $\exp[-Bn_i^2]$ the selected balls are placed in the same randomly chosen neighboring urn, where i is the urn of the selected particles.

One can easily see that the probability that two randomly selected balls belong to the i th urn is given as $(N_i/N)(N_i-1)/(N-1)$, which for $N \rightarrow \infty$ becomes n_i^2 . Multiplying n_i^2 with the transition probability $\exp[-Bn_i^2]$ we obtain that the flux in the pair model is proportional to Eq. (1). It means that as far as the steady-state properties are concerned, the pair model is equivalent to the MWL [6,7]. In particular, for $L=2$ one easily obtains the critical value $B=4$ for the continuous transition between the symmetric ($B < 4$) and asymmetric phase ($B > 4$). For $L=3$ one obtains two critical points $B_1=6.552703411\dots$ and $B_2=9$. The first one can

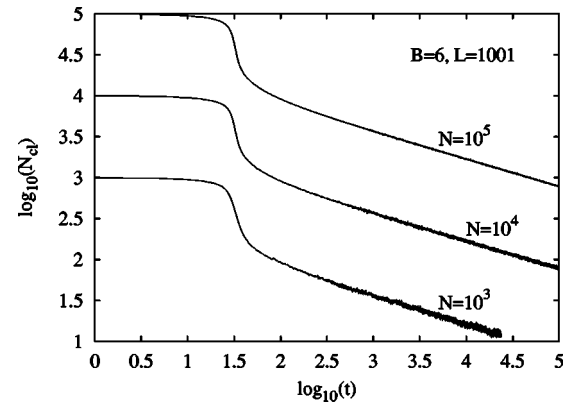


FIG. 6. The average occupancy of a central urn N_{cl} as a function of time t for the pair model. The slope of decay is very close to 1/3 which confirms the anomalous diffusive nature of spreading ($N_{cl} \sim t^{-1/3}$). Each curve is obtained from averaging over 50 independent runs.

only be determined numerically. Similarly to Fig. 1, for $B < B_2$ the symmetric solution is stable whereas for $B > B_1$ the asymmetric solution is stable. In the interval $B \in [B_1, B_2]$ both symmetric and asymmetric solutions are stable, which is the interval showing hysteresis with respect to the driving parameter B .

Qualitatively the dynamical properties of cluster configurations in the pair model are similar to those described in preceding section. In particular, for $L=3$ and $B=B_1$, the average lifetime of a cluster τ as a function of the number of balls N once more shows a power-law divergence $\tau = N^z$, with $z=0.31(3)$ suggesting that $z=1/3$. It shows a certain universality of this exponent with respect to different dynamical rules.

Finally, Fig. 6 shows the diffusion of the broken-down cluster. Since the asymptotic slope of our data is very close to 1/3 we conclude that in this case the diffusion is anomalous, as already predicted by van der Meer *et al.* who used the continuous approach [9].

The pair model and the model examined in the preceding section exhibit qualitatively similar behavior for most of the physical quantities. The main difference is the diffusion: it is anomalous in the pair model and ordinary in model examined in the preceding section. It would be desirable to experimentally examine the nature of diffusion in such systems.

V. CONCLUSIONS

We examined two $L > 2$ versions of the L -urn model of compartmentalization of vibrated sand. Our models qualitatively recover experimental findings and previous steady-state calculations. In addition, our models take into account fluctuations caused by the finite number of balls. Using symmetry properties, we related them with high-dimensional Potts model and argued that for $L > 2$ phase transitions in such systems should be discontinuous. Although several quantities exhibit qualitatively a similar behavior for the two different versions of the model, there are important differ-

ences too. In particular, these models predict a different diffusion of a broken-down cluster, which could be either ordinary or anomalous. It shows that the type of diffusion is very sensitive to dynamical rules of the model, and consequently, to the form of the flux.

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